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Analytical results of the k-core pruning process on multiplex networks

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Multiplex networks are generally considered as networks that have the same set
of vertices but different types of edges. Multiplex networks are especially useful
when describing systems with several kinds of interactions. In this paper, we
study the analytical solution of the **k**-core pruning process on multiplex
networks. *k*-Core decomposition is a widely used method to find the dense
core of the network. Previously, the Non-Backtracking Expansion Branch
(NBEB) has been found to be able to easily derive the exact analytical results
in the *k*-core pruning process. Here, we further extend this method to solve the
k-core pruning process on multiplex networks by designing a variation of the
method called the Multicolor Non-Backtracking Expansion Branch (MNEB). Our
results show that, given any uncorrelated multiplex network, the Multicolor
Non-Backtracking Expansion Branch can offer the exact solution for each
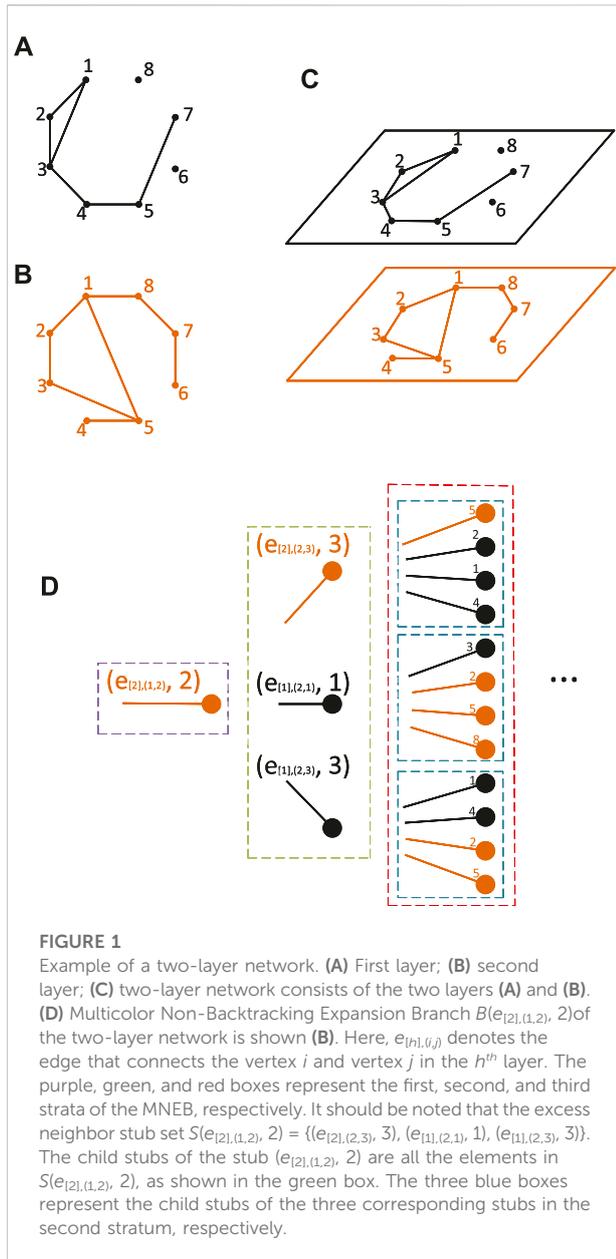intermediate state of the pruning process.

KEYWORDS

complex network, complex system, k-core, multiplex network, network theory

1 Introduction

Graphs are often used to model systems that consist of interacting people or entities,
where the vertices represent people or entities and the edges represent connections.
Nowadays, many graphs are built in this way, from a variety of systems and applications,
such as online social networks, e-commerce platforms, and even protein interaction
networks. One of the most important tasks in analyzing these graphs is to find the densest
part of the network where the vertices are closely related to each other [1–4]. The most
commonly used algorithm for this problem is called the k-core decomposition [5], in
which the goal is to find the subgraph consisting of the vertices that are left after all vertices
whose degrees less than *k* are removed. *k*-Core decomposition is widely used to identify
the important nodes of networks [6–8], help visualize network structures [9,10],
understand and explain the collaborative process in social networks [11,12], describe
protein functions based on protein–protein networks [13,14], and promote network
methods for large text summaries [15].

Previously, many researchers [12,16–18] have focused on solving the *k*-core
decomposition problem on a single-layer network. Wu [19] showed that the Non-
Backtracking Expansion Branch can be used to obtain the exact results of the *k*-core

FIGURE 1
Example of a two-layer network. **(A)** First layer; **(B)** second layer; **(C)** two-layer network consists of the two layers **(A)** and **(B)**. **(D)** Multicolor Non-Backtracking Expansion Branch $B(e_{[2],(1,2)}, 2)$of the two-layer network is shown **(B)**. Here, $e_{[h],(i,j)}$ denotes the edge that connects the vertex i and vertex j in the hth layer. The purple, green, and red boxes represent the first, second, and third strata of the MNEB, respectively. It should be noted that the excess neighbor stub set $S(e_{[2],(1,2)}, 2) = \{(e_{[2],(2,3)}, 3), (e_{[1],(2,1)}, 1), (e_{[1],(2,3)}, 3)\}$. The child stubs of the stub $(e_{[2],(1,2)}, 2)$ are all the elements in $S(e_{[2],(1,2)}, 2)$, as shown in the green box. The three blue boxes represent the child stubs of the three corresponding stubs in the second stratum, respectively.

naturally use multiplex networks [20] that have the same set of vertices but different kinds of edges to represent these systems. Multiplex networks are also called multirelational networks, in which the edges between vertices can represent several different types of interactions. These multiplex networks can be used to describe real networks [21] or dynamic processes [22]. In recent years, there has been a trend in network research to study k-kernels on multiplexed networks. Azimi-Tafreshi [23] derived the self-consistent equation, obtained the birth point and relative size of the k-core of the uncorrelated multiplex networks with arbitrary degree distribution, and found hybrid phase transition. Subsequently, by analyzing different real multiplex networks, Osat [24] proposed a new multiplex network model and showed that interlayer correlations are important in characterizing their k-core structure, that is, the organization in the shell layers of nodes with increasing degrees. Radicchi [25]found that for larger numbers of layers, adding new layers increases the robustness of the system by creating redundant interdependencies between layers. Shang [26] revealed that there is a tipping point in the number of layers beyond which the multilayer k-core suddenly collapses. Boccaletti [20] proposed a comprehensive overview of the organization and dynamics between the components (layers) is given and several related issues are covered, ranging from a complete redefinition of fundamental structural measures to understanding how the multilayered nature of networks affects processes and dynamics. In a multiplex network, each kind of connection is represented by a unique layer, and the same vertex is allowed to have different network structures in different layers. Figures 1A–C show a simple example of a multiplex network.

These existing papers show that both intra-layer and inter-layer correlations have important effects on k-core phase transition. For a long time, due to the lack of mathematical tools, theoretical analysis of the k-core process on multiplex networks has been rare. Here, we show that by assigning different colors to distinguish the types of interactions, we can use the Multicolor Non-Backtracking Expansion Branch (MNEB) method to analytically obtain the results of each step in the **k**-core pruning process in a multiplex network.

pruning process on single-layer networks. The Non-Backtracking Expansion Branch is an alternative representation to the usual adjacency matrix of a network, which is constructed as an infinite tree having the same local structural information as the given network when observed by non-backtracking walkers.

These findings are important for our understanding of the structure of complex networks. Meanwhile, in real-world scenarios, it is common that we have to deal with systems that consist of many different types of interactions. As a result, systems cannot be represented by a single-layer network. For systems with multiple kinds of connections, we

2 Materials and methods

k-Core decomposition on multiplex networks is used to find the largest subgraph in which the degree of each vertex is at least k_i in the ith layer (here **k** is a non-negative integral vector). In the previous paper [23], the researchers gave the analytical result of the final size of the **k**-core on multiplex networks. Here, we show that the Non-Backtracking Expansion Branch (NBEB) method can be used to obtain the complete solution of **k**-core decomposition on the multiplex, in which not only the final

state but also each intermediate state of the pruning process can be obtained analytically.

Given a multiplex network in which each layer is a simple graph, the standard pruning algorithm for **k**-core decomposition is as follows: for a given sequence of k_i, at each step, we remove these nodes whose degrees in any of these layers are less than the corresponding k. In the following, we analyze, in detail, the pruning process and attempt to give the size of the remaining vertices after each step.

First of all, we introduce the definitions of the terms that are used in the remainder of the paper. Suppose a given multiplex network has R layers. For convenience, we assign each layer with a color c_i to distinguish the edges that belong to different network layers. A "stub" is defined as a combination of an edge $e_{[i]}$ and one of its end vertex V, denoted by $(e_{[i]}, V)$, where the subscript $[i]$ here means that it belongs to the ith layer, and i can be any integer in $[1, R]$. Obviously, the stub $(e_{[i]}, V)$ has the color c_i. We denote by k_i the degree of the vertex V in the ith layer. In the ith layer, we define the neighbor stub set of the vertex V: $S_i(V) = \{(e_{[i],1}, V_{[i],1}), (e_{[i],2}, V_{[i],2}), \ldots (e_{[i],k_i}, V_{[i],k_i})\}$; here, $\{V_{[i],1}, V_{[i],2}, \ldots, V_{[i],k_i}\}$ are the neighbors of V in the ith layer, and $\{e_{[i],1}, e_{[i],2}, \ldots, e_{[i],k_i}\}$ are the edges connecting them to V, and the excess neighbor stub set of any stub $(e_{[i]}, V)$ in the ith layer is found to be the complementary set of $\{(e_{[i]}, V^*)\}$ in $S_i(V)$, which is $S_i(e_{[i]}, V) = S_i(V)/\{(e_{[i]}, V^*)\}$ (V^* is the neighbor of V via $e_{[i]}$).

Consequently, for the whole multiplex network, we define the neighbor stub set of the vertex V as

$$S(V) = \bigcup_{j=1}^{R} S_j(V) \qquad (1)$$

and the excess neighbor stub set of $(e_{[i]}, V)$ as

$$S(e_{[i]}, V) = \left(\bigcup_{j=1, j\neq i}^{R} S_j(V) \right) \cup S_i(e_{[i]}, V). \qquad (2)$$

It should be noted that the aforementioned expression of the excess neighbor stub set $S(e_{[i]}, V)$ is equivalent to the complementary set of $\{(e_{[i]}, V^*)\}$ in $S(V)$.

Similar to the definition of the Non-Backtracking Expansion Branch (NBEB) in one layer network, starting from any stub $(e_{[i]}, V)$, we can define such a tree-like structure, which we call the Multicolor Non-Backtracking Expansion Branch(MNEB).

The chosen stub $(e_{[i]}, V)$ (with color c_i) is the root of the MNEB, regarded as the first stratum. For any known n^{th} stratum of the MNEB, we can further find the child stubs of each stub in the nth stratum that are all the elements in its excess neighbor stub set, and all these child stubs constitute the $(n+1)^{th}$ stratum of the MNEB. For example, if we have a stub $(e_{[i]}^*, V^*)$ in the nth stratum, its child stubs are all the stubs that belong to $S(e_{[i]}^*, V^*)$. We can continue this process so that we obtain the MNEB of the stub $(e_{[i]}, V)$, denoted by $B(e_{[i]}, V)$. Obviously, there can be different colored stubs in one MNEB. Figure 1D provides an illustration of how the MNEB is constructed.
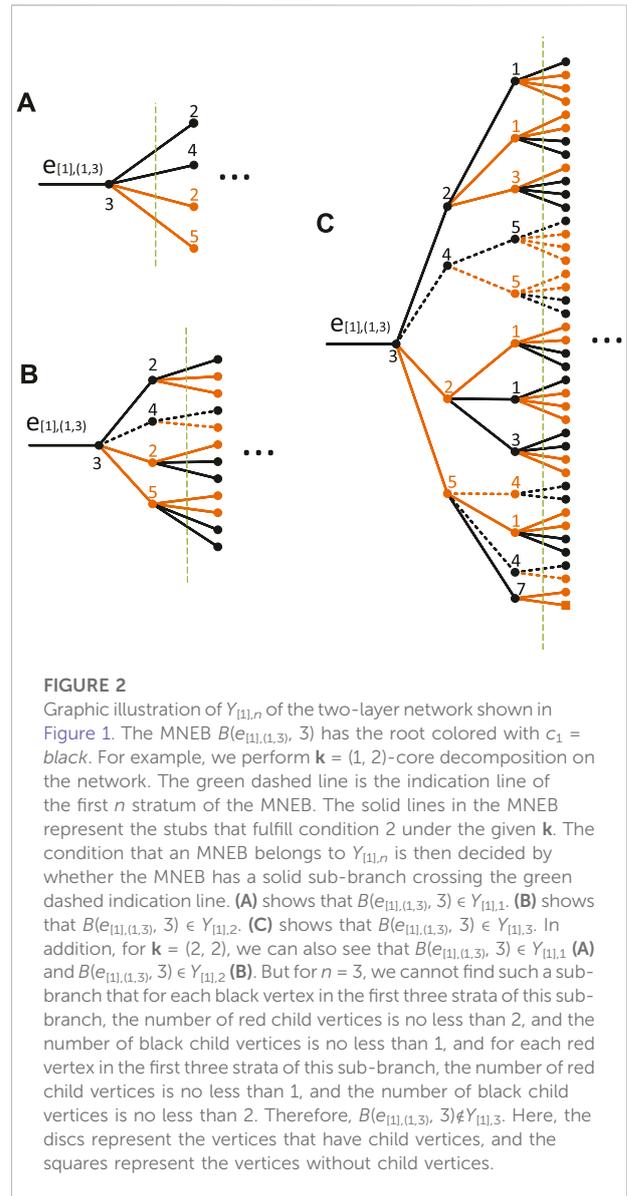

FIGURE 2
Graphic illustration of $Y_{[1],n}$ of the two-layer network shown in Figure 1. The MNEB $B(e_{[1],(1,3)}, 3)$ has the root colored with $c_1 = black$. For example, we perform **k** = (1, 2)-core decomposition on the network. The green dashed line is the indication line of the first n stratum of the MNEB. The solid lines in the MNEB represent the stubs that fulfill condition 2 under the given **k**. The condition that an MNEB belongs to $Y_{[1],n}$ is then decided by whether the MNEB has a solid sub-branch crossing the green dashed indication line. **(A)** shows that $B(e_{[1],(1,3)}, 3) \in Y_{[1],1}$. **(B)** shows that $B(e_{[1],(1,3)}, 3) \in Y_{[1],2}$. **(C)** shows that $B(e_{[1],(1,3)}, 3) \in Y_{[1],3}$. In addition, for **k** = (2, 2), we can also see that $B(e_{[1],(1,3)}, 3) \in Y_{[1],1}$ **(A)** and $B(e_{[1],(1,3)}, 3) \in Y_{[1],2}$ **(B)**. But for $n = 3$, we cannot find such a sub-branch that for each black vertex in the first three strata of this sub-branch, the number of red child vertices is no less than 2, and the number of black child vertices is no less than 1, and for each red vertex in the first three strata of this sub-branch, the number of red child vertices is no less than 1, and the number of black child vertices is no less than 2. Therefore, $B(e_{[1],(1,3)}, 3) \notin Y_{[1],3}$. Here, the discs represent the vertices that have child vertices, and the squares represent the vertices without child vertices.

For a given R-dimensional positive integral vector **k** = $(k_1, k_2, \ldots, k_R)$, we can find a set of MNEBs $Y_{[i],n}$ (n is a positive integer) for each $1 \leq i \leq R$ that meet the following two conditions: 1. the root of the MNEB is colored with c_i; 2. there exists a sub-branch of the MNEB that contains the root stub; for each vertex colored with c_j in the first n strata of this sub-branch, it has at least k_j-1 child vertices colored with c_j and at least k_l child vertices colored with c_l for every $l \neq j (1 \leq l \leq R)$. Figure 2 shows the details of how to decide whether an MNEB belongs to $Y_{[i],n}$. When $n = 0$, for each $1 \leq i \leq R$, we define $Y_{[i],0}$ as all the MNEBs whose roots are colored with c_i. Obviously, $Y_{[i],0} \supset Y_{[i],1} \supset \cdots \supset Y_{[i],n} \supset \ldots$. We denote $S_{MNEB}(V)$ as the set of MNEBs of all the stubs in $S(V)$ and $S_{MNEB}(e_{[i]}, V)$ as the set of MNEBs of all the stubs in $S(e_{[i]}, V)$. It is easy to obtain the following theorem from the definition of $Y_{[i],n}$:

Theorem 1. For a stub $(e_{[i]}, V)$ in the ith layer, the MNEB $B(e_{[i]}, V)$ belongs to $Y_{[i],n}$ if and only if among all the MNEBs in $S_{MNEB}(e_{[i]}, V)$, at least k_i-1 MNEBs belong to $Y_{[i],n-1}$ and k_l MNEBs belong to $Y_{[l],n-1}$ for every $l \neq j$ $(1 \leq l \leq R)$.

Let S_n be the set of the remaining vertices after n^{th} pruning, and the following theorem can be established.

Theorem 2. It is denoted by V, a vertex in the network, $V \in S_n$, if and only if among the MNEBs in $S_{MNEB}(V)$, for every $1 \leq l \leq R$, at least k_l MNEBs belong to $Y_{[l],n-1}$.

The proof of Theorem 2 is given in Appendix. An illustration of the five MNEBs in $S_{MNEB}(1)$ of the network from Figure 1 along with a short exemplary analysis using Theorem 2 is presented in Appendix as well.

3 Analysis on large uncorrelated multiplex networks

As a special case, we start with the partly uncorrelated multiplex networks, in which the degrees of vertices are uncorrelated in each layer but the degrees of vertices in different layers are allowed to be interdependent.

For a random vertex V, it has the degree series $(i_1, i_2, \ldots, i_R)$ on a large uncorrelated multiplex network, where $i_1, i_2, \ldots, i_R$ represent the degrees of a vertex in 1, 2, ..., R layers of the network, respectively. The joint degree distribution probability of the vertex is denoted by $p_{i_1, i_2, \ldots, i_R}$, and the joint excess degree distribution of the vertex in the jth layer is denoted by $q_{i_1, i_2, \ldots, i_R}^{[j]}$, which is the probability that follows a randomly chosen edge in the jth layer, and one of its endpoints has the excess degree k_i, while in the hth layer$(h \neq j)$, its degree is i_h. After that, we can define the following two generating functions:

$$G_0(z_1, z_2, \ldots, z_R) = \sum_{i_1=0}^{\infty} \ldots \sum_{i_R=0}^{\infty} p_{i_1, i_2, \ldots, i_R} z_1^{i_1} \cdot \cdots \cdot z_R^{i_R}, \quad (3)$$

$$G_1^{[j]}(z_1, z_2, \ldots, z_R) = \sum_{i_1=0}^{\infty} \ldots \sum_{i_R=0}^{\infty} q_{i_1, i_2, \ldots, i_R}^{[j]} z_1^{i_1} \cdot \cdots \cdot z_R^{i_R}, \quad (4)$$

where the superscript in the second definition indicates the generating function defined in the jth layer.

These two generating functions are related by

$$G_1^{[j]}(z_1, z_2, \ldots, z_R) = \frac{1}{c_j} \frac{\partial G_0(z_1, z_2, \ldots, z_R)}{\partial z_j}, \quad (5)$$

where c_j is the average degree of the jth layer network. For convenience, we introduce the following denotation:

$$\sum_{\mathbf{i}=\mathbf{x}}^{\mathbf{t}} = \sum_{i_1=x_1}^{t_1} \sum_{i_2=x_2}^{t_2} \ldots \sum_{i_R=x_R}^{t_R},$$

where $\mathbf{x}$ and $\mathbf{t}$ are two fixed integral R dimensional vectors. The aforementioned denotation means to take the sum for $\mathbf{i}$ from the first component to the last component. Of course, there must be $t_j \geq x_j$ for each $1 \leq j \leq R$.

Let $y_{[j],n}$ be the probability that an MNEB whose root is colored with c_j belongs to $Y_{[j],n}$, then from Theorem 1, we can obtain the recursive relationship:

$$\begin{aligned} y_{[j],n} &= \sum_{\mathbf{i}=\mathbf{k}_j}^{\infty} q_{\mathbf{i}}^{[j]} \sum_{\mathbf{m}=\mathbf{k}_j}^{\mathbf{i}} \prod_{l=1}^{R} \binom{i_l}{m_l} (y_{[l],n-1})^{m_l} (1 - y_{[l],n-1})^{i_l-m_l} \\ &= \sum_{\mathbf{m}=\mathbf{k}_j}^{\infty} \frac{\partial^{(m_1+\cdots+m_R)} G_1^{[j]}}{\partial z_1^{m_1} \ldots \partial z_R^{m_R}}\bigg|_{\mathbf{z}=\mathbf{1}-\mathbf{y}_{n-1}} \prod_{l=1}^{R} \frac{(y_{[l],n-1})^{m_l}}{m_l!}, \end{aligned}$$
$$(6)$$

where $y_{[j],0} = 1$ for every $1 \leq j \leq R$. In the aforementioned equation, it should be noted that the summation indexes $\mathbf{i} = (i_1, i_2 \ldots i_R)$ and $\mathbf{m} = (m_1, m_2 \ldots m_R)$ are vectors. When performing a $\mathbf{k}$-core on the multiplex network ($\mathbf{k}$ is a vector), we define an R-dimensional integral vector $\mathbf{k}_j = (k_1, \ldots, k_{j-1}, k_j-1, k_{j+1} \ldots k_R)$. $\mathbf{y}_{n-1}$ denotes the R-dimensional vector $(y_{[1],n-1}, y_{[2],n-1} \cdots y_{[R],n-1})$. Therefore, $\mathbf{1}-\mathbf{y}_{n-1}$ is the R- dimensional vector $(1 - y_{[1],n-1}, 1 - y_{[2],n-1}, \ldots 1 - y_{[R],n-1})$.

Then, we denote by s_n the probability that a randomly chosen vertex belongs to S_n.

$$\begin{aligned} s_n &= \sum_{\mathbf{i}=\mathbf{k}}^{\infty} p_{\mathbf{i}} \sum_{\mathbf{m}=\mathbf{k}}^{\mathbf{i}} \prod_{l=1}^{R} \binom{i_l}{m_l} (y_{[l],n-1})^{m_l} (1 - y_{[l],n-1})^{i_l-m_l} \\ &= \sum_{\mathbf{m}=\mathbf{k}}^{\infty} \frac{\partial^{(m_1+\cdots+m_R)} G_0}{\partial z_1^{m_1} \ldots \partial z_R^{m_R}}\bigg|_{\mathbf{z}=\mathbf{1}-\mathbf{y}_{n-1}} \prod_{l=1}^{R} \frac{(y_{[l],n-1})^{m_l}}{m_l!}. \end{aligned} \quad (7)$$

In the completely uncorrelated multiplex network, where there exists no correlation in different layers, we have $p_{i_1, \ldots, i_R} = p_{i_1} \cdot p_{i_2} \cdots p_{i_R}$; hence, the generating function can be simplified:

$$G_0(z_1, z_2, \ldots, z_R) = \prod_{j=1}^{R} G_{[j],0}(z_j), \quad (8)$$

$$G_1^{[j]}(z_1, z_2, \ldots, z_R) = G_{[j],1}(z_j) \prod_{h=1, h \neq j}^{R} G_{[h],0}(z_h). \quad (9)$$

Here, $G_{[j],0}(z_j)$ and $G_{[j],1}(z_j)$ are the generating functions of the degree distribution and excess degree distribution of the jth layer network, respectively. Substituting these two generating functions into Eqs 6, 7, we obtain

$$\begin{aligned} y_{[j],n} &= \left[1 - \sum_{m=0}^{k_j-2} \frac{(y_{[j],n-1})^m}{m!} G_{[j],1}^{(m)}(1 - y_{[j],n-1}) \right] \\ &\times \prod_{h=1, h \neq j}^{R} \left[1 - \sum_{m=0}^{k_h-1} \frac{(y_{[h],n-1})^m}{m!} G_{[h],0}^{(m)}(1 - y_{[h],n-1}) \right] \end{aligned} \quad (10)$$

and:

$$s_n = \prod_{h=1}^{R} \left[1 - \sum_{m=0}^{k_h-1} \frac{(y_{[h],n-1})^m}{m!} G_{[h],0}^{(m)}(1 - y_{[h],n-1}) \right]. \quad (11)$$
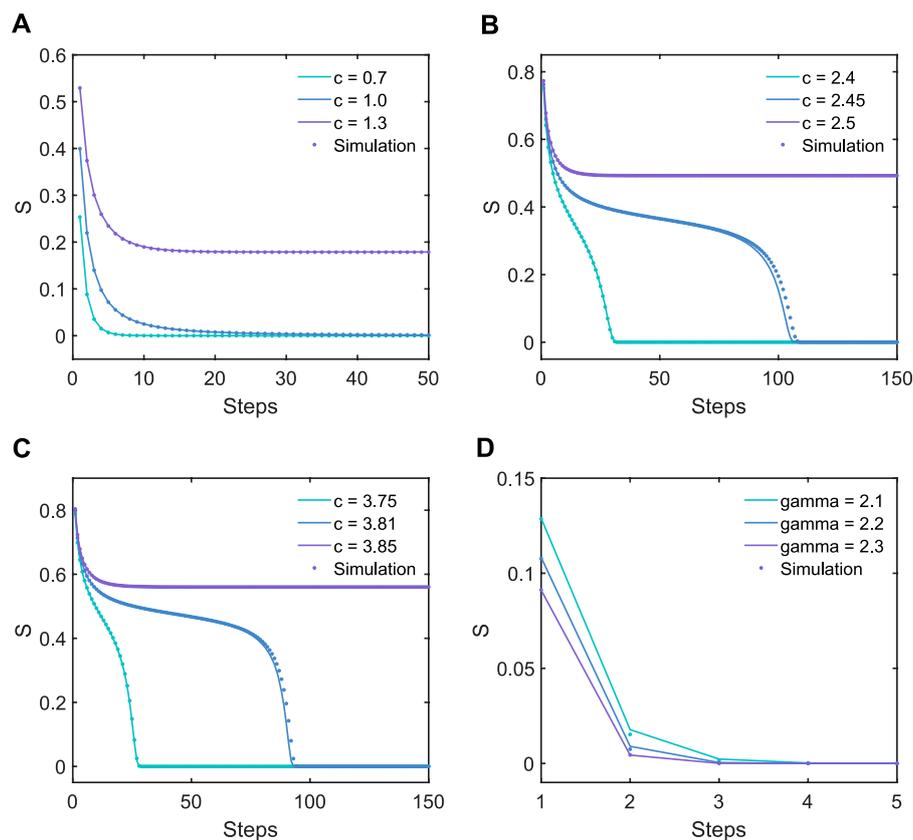
FIGURE 3
Theoretical results (solid lines) and numerical simulation results (dots) for **k**-core decompositions on multiplex networks with identical parameters. All simulations are performed on networks with 10^7 vertices, except that in the simulations of $c = 2.45$ **(B)**, $c = 3.81$ **(C)**, and $\gamma = 2.1$ **(D)**, the networks contain 5×10^7 vertices. **(A)** Results of the (1,1)-core pruning process on three different two-layer uncorrelated ER networks. It should be noted that in this case, it shows a continuous phase transition. **(B)** Results of the (1,1,1)-core pruning process on three different three-layer uncorrelated ER networks. In this case, the networks exhibit a discontinuous phase transition. **(C)** Results of the (2,2)-core pruning process on three different two-layer uncorrelated ER networks. It should be noted that in this case, the networks exhibit a discontinuous phase transition, different from the case shown **(A)**. **(D)** Results of the (2,2)-core pruning process on three different two-layer uncorrelated SF networks. In this case, the (2,2)-core does not exist for $\gamma > 2$.

We further perform several numerical simulations to validate the theoretical results, as given in Section 4.

4 Numerical simulations

To validate our method, we perform several **k**-core decompositions on completely uncorrelated multiplex Erdős–Rényi networks (ER networks) and scale-free networks (SF networks). The ER networks are randomly generated with the parameter c to denote the average degree of vertices, and the order of power–law distribution on SF networks is denoted by γ. In the numerical simulation, we build a multiplex network for the same group of vertices, and different layers of the

network have different interaction topologies. At each pruning step, we examine the changes of all layers at the same time; as long as the degree of a vertex in a certain layer is less than the corresponding k of that layer, this node will be removed from all layers. The results are shown in Figure 3. It can be observed that our theoretical results are in perfect accordance with the numerical simulations. Figures 3A-D show the theoretical results and numerical results on 4 types of pruning process on several networks. Interestingly, we can find that on ER networks, (1,1)-core pruning is a continuous phase transition. Compared to this, Figure 3B shows that in the three-layer ER network, the phase transition of the k-core near the critical point shows a discontinuous phase transition, which is also reflected in [26], in which it is observed that as the number of layers of the network

increases, the system tends to collapse drastically. In addition, as shown in Figure 3C, discontinuous phase transitions also occur in the (2, 2)-core of the two-layer network. The same is shown in Figure 3B; the k-core pruning process also exhibits rich critical behavior when c is slightly lower than the critical point. The system will go through a long-term transient process and finally collapse, forming a discontinuous phase transition.

5 Conclusion

Overall, in this paper, we derive a new variation of the Non-Backtracking Expansion Branch called the Multicolor Non-Backtracking Expansion Branch, specially designed to solve the **k**-core pruning process on multiplex networks. In a multiplex network, each layer of the network is assigned a unique color, and then, the Multicolor Non-Backtracking Expansion Branch is constructed as an infinite tree having the same local structural information as the given multiplex network, when observed by non-backtracking walkers. We find that with this representation, one can obtain the analytical results of the **k**-core pruning process on any given multiplex network, regardless whether the correlation exists or not. The theoretical results obtained by our method are further validated by numerical simulations. Due to the diversity of edges in multiplex networks, the types of interactions that can be expressed will be much richer than those in a single-layer network. Many characteristic behaviors lacking in single-layer networks emerge in multilayer networks. For example, the phase transition of the k-core in a multi-layer network shows a different behavior compared to that of a single-layer network, as shown in Figure 3, and our method can be used for both single-layer and multi-layer networks and can obtain the analytical results of its k-core.

Our method opens new possibilities to analytically solve the **k**-core pruning process on any given multiplex network, which is valuable for both theoretical studies and real-world applications.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

Author contributions

G-YS designed the research; R-JW and G-YS performed the theoretical analysis; Y-XK, G-YS, and Y-CZ designed the figures and performed the numerical experiments. All authors wrote the manuscript.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Appendix

6.1 Proof of Theorem 2

We use mathematical induction to prove the theorem. It is obvious that the theorem holds for $n = 1$. Now, we prove that if the theorem is true for $n-1$, the theorem can be established for n.

First, we prove the sufficiency, that is, for every $1 \leq l \leq R$, when at least k_l MNEBs in $S_{MNEB}(V)$ belong to $Y_{[l],n-1}$, there must be $V \in S_n$. Since for every $1 \leq l \leq R$, $Y_{[l],n-1} \subset Y_{[l],n-2}$, we obtain $V \in S_{n-1}$, and in any given layer (for instance, the ith layer), suppose that $\{B(e_{[i],j_1}, V_{j_1}), \ldots, B(e_{[i],j_m}, V_{j_m})\} \subset S_{MNEB}(V)$ belong to $Y_{[i],n-1}$, here $m \geq k_i$; so for each $1 \leq r \leq m$, in $S_{MNEB}(e_{[i],j_r}, V_{j_r})$, at least k_i-1 MNEBs belong to $Y_{[i],n-2}$, and k_l MNEBs belong to $Y_{[l],n-2}$ for every $l \neq j(1 \leq l \leq R)$. On the other hand, $B(e_{[i],j_r}, V) \in Y_{[i],n} \subset Y_{[i],n-2}$, so in $S_{MNEB}(V_{j_r})$, for every $1 \leq l \leq R$, at least k_l MNEBs belong to $Y_{[l],n-2}$. The induction hypothesis gives $V_{j_r} \in S_{n-1}$. Therefore, in $(n-1)^{th}$ pruning, in any given i^{th} layer, at least k_i neighbors of V are retained. We can conclude that V is still retained in the nth pruning.

Next, we prove the necessity. We attempt to prove that when there exists an l in $[1, R]$, which satisfies that at most k_l-1 MNEBs in $S_{MNEB}(V)$ belong to $Y_{[l],n-1}$, there must be $V \notin S_n$. Since for an MNEB $B(e_{[l],r}, V_r)$ whose root is colored with c_l in $S_{MNEB}(V)$ that does not belong to $Y_{[l],n-1}$, from Theorem 1, we know that in $S_{MNEB}(e_{[l],r}, V_r)$, either at most k_l-2 MNEBs belong to $Y_{[l],n-2}$, or there exists $h \neq l$, $1 \leq h \leq R$ that at most k_h-1 MNEBs belong to $Y_{[h],n-2}$. Therefore, in $S_{MNEB}(V_r)$, there exists $1 \leq h \leq R$ that at most k_h-1 MNEBs belong to $Y_{[h],n-2}$. From the induction hypothesis, we know that $V_r \notin S_{n-1}$, which means after $(n-1)^{th}$ pruning, in the lth layer, at most k_l-1 neighbors of V survived. So either V has been pruned in $(n-1)^{th}$ pruning or even before, or it survived in the $(n-1)^{th}$ pruning but would be deleted in the nth pruning since its remaining neighbors in the lth layer are less than k_l after the $(n-1)^{th}$ pruning; then, we have $V \notin S_n$.

At this point, sufficiency and necessity are proved, and Theorem 2 can be established.

6.2 An example of Theorem 2

Figure A1 are $B(e_{[1],(1,3)}, 3)$, $B(e_{[1],(1,2)}, 2)$, $B(e_{[2],(1,2)}, 2)$, $B(e_{[2],(1,5)}, 5)$, and $B(e_{[2],(1,8)}, 8)$, respectively. For $\mathbf{k} = (2, 1)$

core decomposition, we can find $B(e_{[1],(1,3)}, 3) \in Y_{[1],\infty}$, $B(e_{[1],(1,2)}, 2) \in Y_{[1],\infty}$, $B(e_{[2],(1,2)}, 2) \in Y_{[2],\infty}$, $B(e_{[2],(1,5)}, 5) \in Y_{[2],\infty}$, and $B(e_{[2],(1,8)}, 8) \notin Y_{[2],1}$. So in $S_{MNEB}(1)$, there are two MNEBs that belong to $Y_{[1],\infty}$ and two MNEBs that belong to $Y_{[2],\infty}$. So we have the vertex $1 \in S_\infty$. For $\mathbf{k} = (2, 2)$, we can find that $B(e_{[1],(1,3)}, 3) \in Y_{[1],2}$, but it does not belong to $Y_{[1],3}$. $B(e_{[1],(1,2)}, 2) \in Y_{[1],3}$, but it does not belong to $Y_{[1],4}$. $B(e_{[2],(1,2)}, 2) \in Y_{[2],3}$, but it does not belong to $Y_{[2],4}$. $B(e_{[2],(1,5)}, 5) \in Y_{[2],1}$, but it does not belong to $Y_{[1],2}$. $B(e_{[2],(1,8)}, 8) \notin Y_{[2],1}$. So we can conclude that vertex 1 survives in the first two steps but will be deleted in the third pruning step.

FIGURE A1
All five MNEBs in $S_{MNEB}(1)$ of the network shown in Figure 1.
(A–E) represent the MBEB of e[1],(1,3), e[1],(1,2), e[2],(1,2), e[2],(1,5), e[1],(1,8), respectively.